<u>Mathematical Symmetry</u>

Extra Scene from "The Math Olympian"

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I was enveloped in a warm blanket of air as I walked outside the cafeteria. I squinted as my eyes adjusted to the bright sunlight.

"Smells like summer!" I said to Grace. I looked at the light shining through the tiny blades of grass as we walked towards the picnic area.

"Is here good?" Grace suggested.

"Yeah, here's good."

We both sat down at one of the wooden picnic tables and started to take out our books. We decided to work on our problem set for the day outside after grabbing some lunch.

"Ready to get started?" Grace said.

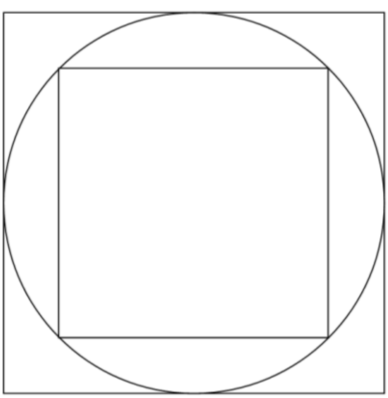
"You got it!" I set out the sheet of problems and the sun lit up the paper.

I'm so happy to be here.

It had been a few days at the math camp and Grace and I got in to the habit of spending as much time as possible outside, taking advantage of the summer warmth. We got right to work and looked at the first problem.

<u>Question #1</u>

A square is inscribed in a circle, and this circle is inscribed in a square. What is the ratio of the areas of the two squares?

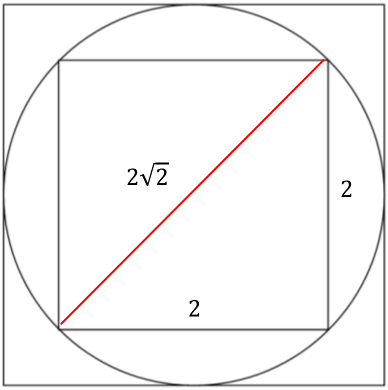


Grace and I both took a moment to think about the problem. Grace started to twirl her pencil, which she always did whenever she had an idea.

"Hey, I think I know how to solve this."

"How?" I asked.

"Pythagorean theorem!"



"Okay, so if we look at this diagram here and make the sides of the smaller square equal to 2, the diameter of the circle has to be $2\sqrt{2}$, so the side length of the larger square has to be $2\sqrt{2}$."

"Why?" I asked.

"Because the diameter of the circle is $2\sqrt{2}$, and since the circle is inscribed in the larger square this has to also be the side length of the larger square."

I paused. "Yeah, I get it."

"We can find the area of the larger square and compare it with the area of the smaller square. The large square has an area of 8 and the smaller square has an area of 4, so the ratio is $2:1$."

Grace continued spin her pencil around her fingers as she finished her explanation.

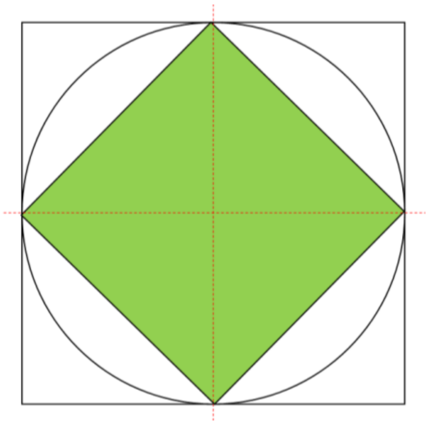
"Does that make sense?" she asked.

I suddenly had an idea for a different way to solve the problem.

"Yeah, it totally does, but I can do it faster. What if we *rotate* the smaller square?"

"What do you mean?"

“Rotate the inner square 45° like this, and divide the square into four equal quadrants…”



I explained to Grace that each quadrant could be split into two congruent triangles. Letting each triangle have area 1, Grace and I could see why the small square has area 4 and the large square has area 8. So, just as before, the ratio of the areas has to be 2:1.

Grace started to twirl her pencil again, thinking about this new way of solving the question.

“Wow, yeah I totally get it. That’s so much faster! How did you see that?”

“You’re always flipping that thing,” I said, pointing to her hand.

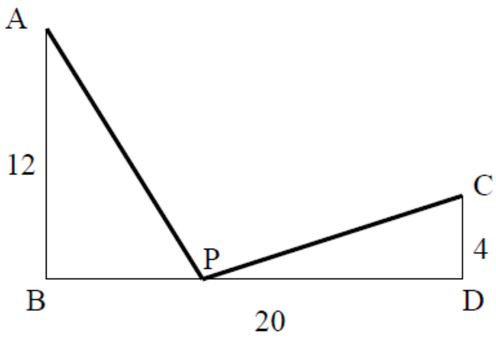
Grace laughed and put down her pencil. “Well, guess I can take credit for this one.”

“Ha ha, very funny. Next question?”

We took a look at the second question. Grace picked up her pencil and began twirling it again.

Question #2

A farmer wants to bring water to his cow from the river, but he’s already had a long day of work and wants to walk as little as possible. Find the point P on the river that minimizes the sum AP + PC.



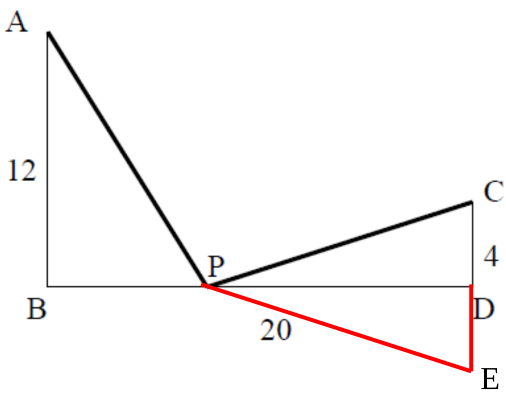
After a few minutes of us staring at the diagram, I still didn't have any good ideas for solving the problem. So Grace was, once again, the first to speak.

"Hey, I totally know how to solve this."

"Great, because I don't get it."

"Okay, so actually my solution is thanks to you, because rotating part of the diagram can help us see the farmer's shortest route."

Grace added two red lines to the diagram.



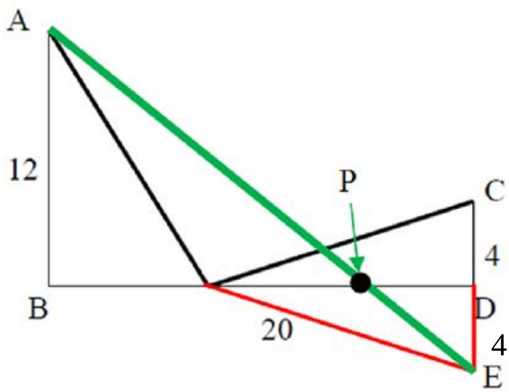
"We want to minimize AP + PC. We can do this with rotation, well not really rotation, but reflection. If we reflect triangle PDC across the horizontal line, and call this new triangle PDE, then we'll be able to find the shortest path."

"Why?" I asked.

"Here's why: let's pretend we're trying to get to E instead of C, and we can do this because for any point P on the horizontal line, PC must equal PE."

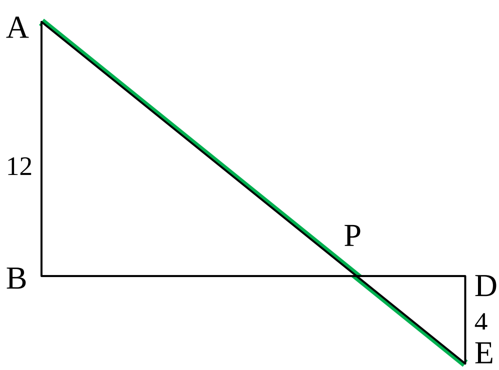
"Oh, I get it!" I said. "For any point P on the horizontal line, AP + PC equals AP + PE. A is fixed, and because E is just the mirror image of C, point E is fixed too. So, the shortest route is the straight line from A to E. This happens when P is directly on the line AE."

Grace marked the desired point P in the diagram.



"Oh cool," she said. "This green line creates two similar triangles: PBA and PDE."

I nodded, seeing why the two triangles were similar. Happy to be up to speed with the problem, I drew a fresh picture that included only the information we needed to solve the problem.

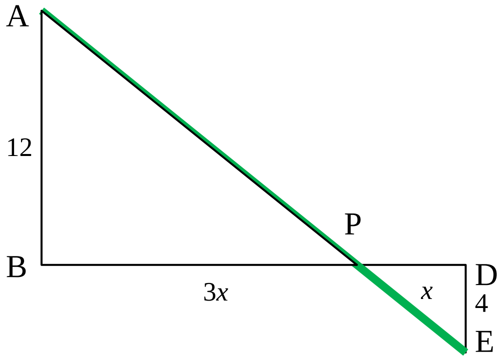


“We want to figure out how far P is away from B and D,” I said.

“Exactly,” said Grace. “But that’s easy. We know that triangles PBA and PDE are similar, so if side AB is three times side DE, then the length of BP must be three times the length of PD.”

I saw the next step in solving the problem and jumped in.

“So then we get the equation $3x + x = 20$, since we know that BD is 20. That means $x = 5$, putting point P exactly 15 meters right of B and 5 meters left of D. Done.”



“Nice,” said Grace, giving me a high-five.

I thought to myself how interesting it was to see that changing certain parts of the diagrams could solve these two problems. It made it so much easier to visualize the solution, and made me think of the importance of perspective in finding an elegant solution.

As we turned to look at the next problem, I wondered if we could use rotations or reflections to solve this one too.

Question #3

Card trick: If you shuffle a deck of cards and then split the deck exactly in half, you will always have the same number of black cards in the first pile as red cards in the second pile. Explain why this trick works.

After reading the problem, I immediately got up and went to get a deck of cards, an essential prop at every math camp. I came back to the picnic table a minute later.

“Mr. Collins always told me that using props helps you understand a problem. Here Grace, can you shuffle the cards?”

“Okay,” Grace answered, taking the cards.

When Grace was done shuffling, I split the deck into two piles of 26. We proceeded to count the number of black and red cards in the first pile, and the number of black and red cards in the second pile.

“It works!” exclaimed Grace. “The number of black cards in the first pile is the same number of red cards in the second pile. I get 10 black cards in the first pile and 10 red cards in the second pile. Also, there are 16 red cards in the first pile and 16 black cards in the second pile.”

“I think I know how this works.” I said. “There are 26 cards per pile. We can say that in the first pile the number of black cards is x and the number of red cards is 26-x. Since the total number of red cards is 26, if there are 26-x red cards in pile one then there must be x red cards in pile two. That’s why the number of black cards in the first pile must equal the number of red cards in the second pile.”

“You lost me,” said Grace. “I still don’t really understand how this works. Can you explain it differently?”

I thought about it for a few moments until I had an idea.

“I’m going to draw a table,” I said. “The rows represent the number of cards for each colour and the columns represent the number of cards in each pile.”

I filled in x in the top-left corner.

	First Pile	**Second Pile**
Black Cards	x	
Red Cards		

Grace jumped in and filled in the remaining squares.

	First Pile	**Second Pile**
Black Cards	x	26-x
Red Cards	26-x	x

"I totally see it," said Grace. "Awesome. Each row and each column adds up to 26. There must be 26 cards in both piles and there must be 26 red cards and 26 black cards. No matter what x is, the number of black cards in the first pile has to be the number of red cards in the second pile."

"Yeah, that's exactly it." I replied. "What a cool problem. I'm going to show Mr. Collins this card trick and see if he can find the math behind it too."

Grace nodded. "I'm going to show this trick to my dad, but I think I'll keep the math a secret and call it magic. So, what's the next question?"

Question #4

Matt the Mathematician is holding two glasses, which are identical in size. He fills glass 1 with water so that it is half full, and he fills glass 2 with oil so that it is also half full. He then takes a teaspoon of oil from glass 2 and pours it into glass 1. Then he mixes up glass 1 a bit, and pours one teaspoon of the that mixture into glass 2. Now there is a little bit of oil in glass 1, and a little bit of water in glass 2. Matt the Mathematician says that there is more oil in glass 1 than there is water in glass 2. Is he correct?

We both paused and took a few minutes to ponder the question.

"This question seems a lot like the one we just did," I said.

"Yeah... it does. If we look at the two glasses as the two piles, and molecules of liquid of oil and water as the black cards and red cards…" said Grace.

"You're right!" I shouted, interrupting. "Based on what we discovered from the card trick, we know that there is the same amount of water in the first glass as there is oil in the second glass."

"Why?" Asked Grace.

"Same idea as before," I said, drawing a table. "If there are x molecules of oil in the first glass, then there must be x molecules of water in the second glass. Just replace 26 with some arbitrary number n representing the number of molecules in each glass."

	First Glass	Second Glass
Oil	x	$n-x$
Water	$n-x$	x

Grace nodded. "Yeah, I totally get it. It's the exact same thing."

I smiled. "So Matt the Mathematician is wrong. There is the same amount of oil in glass 1 as there is water in glass 2."

Just then, Rachel came by and asked us how we were doing.

We both answered at the same time. "We got the first four questions!"

"That's great," said Rachel smiling. "Have you noticed anything special about those questions?"

"Well, I was thinking that our *perspective* made a big difference in finding the solutions," I said.

"Yes," she nodded. "That's it! There's a specific perspective with which you found the solution to these questions, and that perspective is *symmetry*."

"Symmetry?" asked Grace.

"Most certainly," said Rachel. "The first question involved rotational symmetry; the second question involved reflective symmetry; and the last two questions involved algebraic symmetry. Do you see that?"

"Totally," I said.

"In fact, everything has symmetry! You should take a look at some of the work done by a famous artist named M.C. Escher if you want to see some really cool symmetry. You wouldn't know it by looking, but the symmetry of his drawings can be explained using group theory, or abstract algebra, which is what I'm doing my research on! Here, let me show you!"

Grace and I turned to each other and smiled. We were so happy to be here learning about a subject that finds its way into, and expresses itself so uniquely in, every aspect of our world.