Solutions Manual

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Contents

Preface i

Errata for Heavenly Mathematics ii

List of Contributors iii

1 Chapter 1 Solutions 1
2 Chapter 2 Solutions 2
3 Chapter 3 Solutions 5
4 Chapter 4 Solutions 8
5 Chapter 5 Solutions 11
6 Chapter 6 Solutions 15
Preface

Welcome to the solutions manual for the book *Heavenly Mathematics*. Its purpose is to aid the reader through some of the exercises from the book. This manual assumes a solid basis in planar trigonometry as well as fluency in algebra; readers are still strongly encouraged to attempt the problems themselves before resorting to this manual.

This is a student-led project and an online book, which we will update frequently as we add solutions to more problems. Students and former students of the course “Spherical Trigonometry” at Quest University Canada are writing the solutions. Because of this, the solutions might be updated in any order. Solutions from readers are encouraged and may be submitted to the editors for possible inclusion in this manual.

Not all of the exercises in the book will have solutions in this manual. *Heavenly Mathematics* is being used as a textbook for a course; thus some problems must remain unsolved here so they can be assigned as homework. The exercises that will never have solutions here are:

- Chapter 1: 1, 2, 6, 7, 13, 14
- Chapter 2: 3, 4, 5, 6, 7, 8, 10
- Chapter 3: 3, 4, 5, 6, 7
- Chapter 4: 1, 2, 7, 8
- Chapter 5: 3, 4abc, 5a, 6a, 7, 8, 12, 13
- Chapter 6: 1ab, 6, 8, 9, 10, 11, 12

We hope you find these solutions helpful. If you have any suggestions or corrections please let us know.

-Adam Parke and Thomas Turner, editors
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Errata for *Heavenly Mathematics*

**Page 40:** Exercise #12 refers to a spherical polygon with \( n \) sides “(each 180°).” It should read: “(each < 180°).”

**Page 71:** Exercise #7 should refer to exercise #6 from chapter 3, and exercise #8 should refer to exercise #7 from chapter 3.

**Page 80:** The equation
\[
c^2 = a^2 + b^2 + \frac{a^2b^2}{2}
\]
should be
\[
c^2 = a^2 + b^2 - \frac{a^2b^2}{2}.
\]

**Page 105:** The phrase “...and so \( c = 28^\circ \), equal to 1680 nautical miles or 1933 statute miles.” should read “...and so \( c = 56^\circ \), equal to 3360 nautical miles or 2866 statute miles.”
List of Contributors

Contributors are identified in the solutions by their initials.

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12) Label the intersection point on $AB$ as $D$. Draw a horizontal line rightward from $C$ and drop a perpendicular line from $B$ onto it defining $Y$. Since $\angle ACY$ is a right angle, $\angle ACB = 85^\circ$; and $\angle ACD = 90^\circ - 43.867^\circ = 46.133^\circ$.
Then $\angle DCB = \angle ACB - \angle ACD = 38.867^\circ$.
Furthermore since we know the altitude of the mountain and the altitude of the airplane, we have $AC = 8460\ ft - 4135\ ft = 4325\ ft$, thus

$$\sin 43.867^\circ = \frac{CD}{4325} \Rightarrow CD = 2997.167\ ft.$$  

Also

$$\cos 38.867^\circ = \frac{2996.167}{CB} \Rightarrow CB = 3848.36\ ft.$$  

Lastly,

$$\sin 5^\circ = \frac{BY}{3848.36} \Rightarrow BY = 335.4\ ft.$$  

Then the height of the mountain $B$ is $4135\ ft + 335.4\ ft = 4470.4\ ft$. [SH]
Chapter 2 Solutions

1) a) To find eccentricity $EC$, we first find the lengths of the arcs in terms of degrees. For spring,

\[ \frac{94.5}{s} = \frac{365.25}{360} \Rightarrow s = 93.142^\circ, \]

and summer is

\[ \frac{92.5}{q} = \frac{365.25}{360} \Rightarrow q = 91.170^\circ. \]

Figure 2.1: From figure 2.5 in *Heavenly Mathematics* with the apogee marked $\eta$.

In Figure 2.1, arc length for spring, $s$, is $93.17^\circ = 90^\circ + \theta + \phi$. The arc length for summer, $q$, is $91.142^\circ = 90^\circ - \phi + \theta$. Simplifying, we find:

\[ 3.142^\circ = \theta + \phi \]  
\[ \theta = 1.17^\circ + \phi. \]

Substituting (2) into (1) gives

\[ 3.142^\circ = 1.17^\circ + \phi + \phi, \]

simplifying, we find $\phi = 0.986^\circ$. Using this value in (1) gives $\theta = 2.156^\circ$.

Now we find the lengths of the sides of the rectangle containing $C$ and $E$. Because $CA = 1$,

\[ \sin \theta = CD = 0.03762, \text{ and} \]
\[ \sin \phi = ED = 0.01721. \]

The Pythagorean Theorem tells us
\[ CE = \sqrt{ED^2 + CD^2} = \sqrt{0.000296 + 0.00142} = 0.04137. \]

b) Extend \( EC \) to the edge of the circle at \( \eta \) furthest from Earth. The angle between \( C\eta \) and \( Cx \) is the same as \( \angle CED \), so
\[ \sin \angle CED = \frac{CD}{CE} = .9131 \Rightarrow \angle CED = 65.42^\circ. \]

The arc \( \widehat{\eta A} \) is
\[ \angle CED + \theta = 65.42^\circ + 2.156^\circ = 67.572^\circ. \quad [AP], [TT] \]

2) The lune’s surface area is
\[ \frac{\theta}{360^\circ} \cdot (\text{Surface area of unit sphere}). \]

Since the surface area of the unit sphere is \( 4\pi r \),
\[ \frac{\theta}{360^\circ} \cdot 4\pi r^2 = \frac{\theta 4\pi r^2}{360^\circ} = \frac{\theta r^2}{90^\circ}. \quad [TT], [AP] \]
12) An $n$-sided spherical polygon can be divided into $n$ triangles by foining vertices to a central point within the polygon. Since we know that the total angle measurement in a spherical triangle must be $> 180^\circ$, for $n$ triangles the angle sum will be $> 180n^\circ$. We are only after the angles of the polygon, not the total angles of the triangles, so we remove the angles surrounding the central point (which sum to $360^\circ$) from the total sum. This leaves the angle sum $> 180n^\circ - 360^\circ$. \[\text{[AP],[TT]}\]

Figure 2.2: An $n$-sided polygon on a sphere.
Chapter 3 Solutions

1) a) Assign $\lambda = 42^\circ$ and use Menelaus’s Theorems.

To find the right ascension, apply Theorem A to Figure 3.1:

$$\frac{\sin AB}{\sin \gamma A} = \frac{\sin BN \cdot \sin C \varpi}{\sin C N \cdot \sin \varpi \gamma}.$$ 

This gives

$$\frac{\sin(90^\circ - \alpha)}{\sin \alpha} = \frac{\sin 90^\circ}{\sin(90^\circ - \epsilon)} \cdot \frac{\sin(90 - \lambda)}{\sin \lambda}.$$ 

Simplifying, we have

$$\cot \alpha = \frac{1}{\cos \epsilon} \cdot \cot \lambda \Rightarrow \tan \alpha = \cos \epsilon \tan \lambda.$$ 

Using known values we find

$$\alpha = \tan^{-1} (\cos 23.4^\circ \cdot \tan 42^\circ) = 39.57^\circ.$$
To find the declination, we apply Theorem B to Figure 3.1:

\[
\frac{\sin B \Upsilon}{\sin BA} = \frac{\sin BC \sin N \Upsilon}{\sin C \Upsilon \sin NA}.
\]

This gives

\[
\frac{\sin 90^\circ}{\sin(90^\circ - \alpha)} = \frac{\sin 90^\circ}{\sin(90 - \lambda)} \cdot \frac{\sin(90^\circ - \delta)}{\sin 90^\circ}.
\]

Simplifying, we have

\[
\frac{1}{\cos \alpha} = \frac{1}{\cos \lambda} \cdot \cos \delta \Rightarrow \cos \delta = \frac{\cos \lambda}{\cos \alpha}.
\]

Using known values, we find

\[
\delta = \cos^{-1} \left( \frac{\cos 42^\circ}{\cos 39.57^\circ} \right) = 15.41^\circ.\ [TT], [AP]
\]

b) Using Menelaus’ Theorem B

\[
\frac{\sin 90^\circ}{\sin(90^\circ - \alpha)} = \frac{\sin 90^\circ}{\sin(90 - \lambda)} \cdot \frac{\sin(90^\circ - \delta)}{\sin 90^\circ}
\]

Solve for \(\lambda\)

\[
\lambda = \cos^{-1} (\cos \alpha \cos \delta).
\]

Substituting \(\alpha = 126.31^\circ\) and \(\delta = 19.22^\circ\) we find \(\lambda = 123.99^\circ\), and from Appendix A we know that this value corresponds to July 28th.

2) Using the steps from page 54 of *Heavenly Mathematics* we solve for \(\delta, \eta, n, \theta\). We are given \(\epsilon = 23.4^\circ, \phi = 49.3^\circ\) and from Appendix A we find \(\lambda = 58.3^\circ\).

a) To solve for \(\delta\) we use conjunction on the Menelaus configuration \(NGZM\Upsilon\Upsilon\)

\[
\sin \delta = \sin \lambda \sin \epsilon
\]

to determine

\[
\delta = \sin^{-1} (\sin 58.3^\circ \sin 23.4^\circ) = 19.75^\circ.
\]

b) To solve for \(\eta\) we use conjunction on the Menelaus configuration \(NHQME\Upsilon\)

\[
\sin \eta = \frac{\sin \delta}{\cos \phi}.
\]

Using this we substitute in our values and solve for \(\eta\)

\[
\eta = \sin^{-1} \left( \frac{\sin 19.75^\circ}{\cos 49.3^\circ} \right) = 31.21^\circ.
\]
c) To solve for \( n \) we use Menelaus configuration \( NHQME \) with conjunction, but the arc assignment changed

\[
\sin MQ = \frac{\cos \eta}{\cos \delta}
\]

where \( MQ = 90^\circ - n \). We find

\[
n = \cos^{-1} \left( \frac{\cos 31.21^\circ}{\cos 19.75^\circ} \right) = 24.67^\circ.
\]

(1)

d) We find \( MZ \) using Menelaus configuration \( NGZM \) and altering the arc assignment from a)

\[
\sin MZ = \frac{\cos \lambda}{\cos \delta}.
\]

Substituting; we find

\[
MZ = \sin^{-1} \left( \frac{\cos 58.3^\circ}{\cos 19.75^\circ} \right) = 33.93^\circ.
\]

(2)

Combining (1) and (2) we find

\[
\theta = ZQ = MQ - MZ = (90^\circ - 24.67^\circ) - 33.93^\circ = 31.4^\circ. \quad [AP], [TT]
\]
Chapter 4 Solutions

3)

Figure 4.1: Abū Nasr’s figure (Figure 4.1 in *Heavently Mathematics*).

Apply Menelaus conjunction to Figure 4.1; we get

$$\frac{\sin AC}{\sin CB} = \frac{\sin EA}{\sin DE} \cdot \frac{\sin FD}{\sin FB}.$$  

Apply Menelaus disjunction to the same figure; we get

$$\frac{\sin CB}{\sin AB} = \frac{\sin FC}{\sin EF} \cdot \frac{\sin DE}{\sin AD}.$$  

Solving both results for $\frac{\sin CB}{\sin DE}$,

$$\frac{\sin CB}{\sin DE} = \frac{\sin FC}{\sin EF} \cdot \frac{\sin AB}{\sin AD} = \frac{\sin AC}{\sin EA} \cdot \frac{\sin FB}{\sin DF}.$$  

Setting solved conjunction and disjunction equal, we get

$$\frac{\sin FC \cdot \sin AB}{\sin EF \cdot \sin AD} = \frac{\sin AC \cdot \sin FB}{\sin EA \cdot \sin DF} \quad (1)$$  

Since $\triangle FBC$ and $\triangle ACE$ from Figure 4.1 both have two $90^\circ$ angles,

$$AC = FC = EA = FB = 90^\circ.$$  \hspace{1cm} (2)

Substituting (2) into (1) and simplifying gives us

$$\frac{\sin DF}{\sin EF} = \frac{\sin AD}{\sin AB}.$$  \hspace{1cm} $\square [TT], [AP]$

6)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure42.png}
\caption{Indian approach to finding arcs of the ecliptic (Figure 4.4 in *Heavenly Mathematics*).}
\end{figure}

In Figure 4.2, $\angle AO \alpha = \alpha$, $\angle OD \delta = \delta$, and $\angle CO \epsilon = \epsilon$. Therefore,

$$\sin \alpha = \frac{ED}{OD},$$  \hspace{1cm} (1)

$$\sin \delta = \pi D, \cos \delta = OD,$$  \hspace{1cm} (2)

and

$$\sin \epsilon = CK, \cos \epsilon = OK.$$  \hspace{1cm} (3)

We also know $\triangle ED \pi \sim \triangle CO \epsilon$, so

$$\frac{E \pi}{OC} = \frac{D \pi}{CK} = \frac{ED}{OK}.$$  \hspace{1cm} (4)

We also found on page 65 that

$$\sin \delta = \sin \lambda \sin \epsilon.$$  \hspace{1cm} (5)

Substituting (2) into (5) gives $D \pi = \sin \lambda \sin \epsilon$. Using this along with (3) and (4), we find

$$\frac{D \pi}{CK} = \frac{\sin \epsilon \sin \lambda}{\sin \epsilon} = \sin \lambda = \frac{ED}{OK} \Rightarrow ED = \sin \lambda \cos \epsilon.$$

\[
\begin{align*}
\end{align*}
\]
Combining this with (1) and (2), we find the Indian theorem for the right ascension:

\[ \sin \alpha = \frac{\sin \lambda \cos \epsilon}{\cos \delta} \square \ [AP], [TT] \]

9) a) On construction \( PGCBF \), we know \( \widehat{PB} = \widehat{PC} = 90^\circ \). We are also given \( \widehat{BC} = 27.37^\circ \), \( \widehat{GC} = 33.58^\circ \). From these values we can use the Rule of Four Quantities to find \( \widehat{FG} \):

\[
\frac{\sin FG}{\sin BC} = \frac{\sin 90^\circ - \widehat{GC}}{\sin PC} \Rightarrow \sin FG = \cos GC \sin \widehat{BC}.
\]

Substituting in known values:

\[
\sin FG = \cos 33.58^\circ \sin 27.37^\circ = 0.880117;
\]

\( \therefore \widehat{FG} = 22.52^\circ \).

b) Using construction \( EFPNH \) we now solve for \( \widehat{PE} \). In Figure 4, \( \widehat{EF} = \widehat{HE} = 90^\circ \). We also know that \( \widehat{PN} = 90^\circ - \widehat{PG} \). Because \( \widehat{PG} = 90^\circ - \widehat{GC}, \widehat{PN} = \widehat{GC} \). We also know \( \widehat{PE} = 90^\circ + \widehat{FP}, \) so

\[
\frac{\sin(90^\circ - \widehat{FG})}{\sin 90^\circ} = \frac{\sin \widehat{PN}}{\sin \widehat{PE}} \Rightarrow \cos \widehat{FG} = \frac{\sin \widehat{PN}}{\sin(90^\circ + \widehat{FP})} \Rightarrow \cos \widehat{FP} = \sin \widehat{PN} \cos \widehat{FG}.
\]

Substituting known values gives

\[
\cos \widehat{FP} = \frac{\sin 33.58^\circ}{\cos 22.52^\circ} = 0.59876;
\]

\( \therefore \widehat{FP} = 53.21^\circ \Rightarrow \widehat{PE} = 90^\circ + 53.21^\circ = 143.21^\circ. \)

From figure 6, \( \widehat{PB} = \widehat{EF} = 90^\circ \); to determine \( \widehat{ME} \) we need to find \( \widehat{FB} \) and \( \widehat{FM} \). First find \( \widehat{FB} \)

\[ \widehat{FB} = \widehat{PB} - \widehat{PF} = 36.79^\circ \]

We can use \( \widehat{FB} \) and \( \widehat{MB} = 21.67^\circ \) to determine \( \widehat{FM} \):

\[ \widehat{FM} = \widehat{FB} - \widehat{MB} = 15.12^\circ. \]

Since \( \widehat{ME} = 90^\circ - \widehat{FM} \) we have

\[ \widehat{ME} = \widehat{EF} - \widehat{FM} = 74.88^\circ. \]

c) To find \( \widehat{MD} \), apply the Rule of Four Quantities on construction \( EMFDH \):

\[
\frac{\sin \widehat{MD}}{\sin \widehat{FH}} = \frac{\sin \widehat{ME}}{\sin \widehat{EF}} \Rightarrow \widehat{MD} = 63.1^\circ.
\]

d) To find the qibla simply apply the spherical Law of Sines to \( \triangle MPG \). From figure 6, \( \angle GPM = \widehat{BC} = 27.37^\circ, \widehat{MG} = 90^\circ - \widehat{MD}, \) and \( \widehat{PM} = \widehat{PF} + \widehat{FM} = 68.33^\circ; \) so

\[
\frac{\sin \widehat{MG}}{\sin \angle GPM} = \frac{\sin \widehat{PM}}{\sin \angle MGP} \Rightarrow \angle MGP = \sin^{-1} \left( \frac{\sin(68.37^\circ) \sin(27.37^\circ)}{\cos(63.1^\circ)} \right) = 70.83^\circ. \quad [TT], [AP] \]
Chapter 5 Solutions

1) 

Figure 5.1: Tetrahedron ODEF formed between spherical triangle ABC and center point O (similar to Figure 5.2 in Heavenly Mathematics).

From the above figure,

\[
\sin b = \frac{EF}{OE} = \frac{EF}{ED} \cdot \frac{ED}{OE} = \cot A \tan a.
\]
\[
\cos A = \frac{EF}{DF} = \frac{EF}{OF} \cdot \frac{OF}{DF} = \tan b \cot c.
\]
\[
\cos c = \frac{OF}{OD} = \frac{OF}{ED} \cdot \frac{ED}{OD} = \cos b \cos a. \quad \square \quad [AP], [TT]
\]

2) To derive \( \cos c = \cot A \cot B \) we start with \( \cos c = \cos a \cos b \). We solve

\[
\sin a = \tan b \cot B \quad \text{and} \quad \sin b = \tan a \cot A
\]

for \( \cos a \),

\[
\sin a = \tan b \cot B \Rightarrow \cos b = \frac{\sin b}{\sin a} \cdot \cot B
\]

and \( \cos b \),

\[
\sin b = \tan a \cot A \Rightarrow \cos a = \frac{\sin a}{\sin b} \cdot \cot A.
\]
5. CHAPTER 5 SOLUTIONS

Substituting our new values for $\cos a$ and $\cos b$, we arrive at

$$\cos c = \cot A \cot B. \square$$

To derive $\cos A = \cos a \sin B$ we start with $\cos A = \tan b \cot c$, which we rewrite as

$$\cos A = \frac{\sin b}{\sin c} \cdot \frac{\cos c}{\cos b}. \tag{1}$$

Using the identities

$$\sin b = \sin B \sin c$$

and

$$\cos c = \cos a \cos b$$

we solve for $\frac{\sin b}{\sin c}$ and $\frac{\cos c}{\cos b}$. This gives

$$\frac{\sin b}{\sin c} = \sin B \text{ and } \frac{\cos c}{\cos b} = \cos a. \tag{2}$$

Substituting (2) into (1) we have

$$\cos A = \cos a \sin B. \square$$

To derive $\cos B = \cos b \sin A$ we rewrite the formula $\cos B = \tan a \cot c$ as follows:

$$\cos B = \frac{\sin a}{\sin c} \cdot \frac{\cos c}{\cos a}. \tag{3}$$

Using the identities

$$\cos c = \cos a \cos b$$

and

$$\sin a = \sin A \sin c$$

we solve for $\frac{\sin a}{\sin c}$ and $\frac{\cos c}{\cos a}$:

$$\frac{\sin a}{\sin c} = \sin A \text{ and } \frac{\cos c}{\cos a} = \cos b. \tag{4}$$

Substituting (4) into (3), we have

$$\cos B = \cos b \sin A. \square \ [TT], \ [AP]$$
4) **d)** Given the data \( a = 69.72^\circ \) and \( c = 78.42^\circ \), we can find the rest of the right triangle, starting with side \( b \) using

\[
\cos c = \cos a \cos b \Rightarrow \cos b = \frac{\cos c}{\cos a}
\]

So

\[
\cos b = \frac{\cos 78.42^\circ}{\cos 69.72^\circ} \Rightarrow \cos b = 0.57914
\]

\[\Rightarrow b = 54.6097^\circ.\]

We find \( \angle B \) using

\[\cos B = \tan a \cot c.\]

So

\[\cos B = \tan 69.72^\circ \cot 78.42^\circ \Rightarrow \cos B = 0.5545297899 \]

\[\Rightarrow B = 56.3220^\circ.\]

Lastly we find \( \angle A \) using

\[\cos A = \sin B \cos a.\]

So

\[\cos A = \sin 56.32^\circ \cos 69.72^\circ \Rightarrow \cos A = 0.28843 \]

\[\Rightarrow A = 73.24^\circ.\]

**e)** Given \( \angle A = 52.4^\circ \) and \( \angle B = 122.27^\circ \), we begin with

\[\cos c = \cot B \cot A.\]

So

\[\cos c = \cot 122.27^\circ \cot 52.4^\circ \Rightarrow \cos c = -0.48628 \]

\[\Rightarrow c = 119.0961^\circ.\]

Next we find \( b \) using

\[\cos b = \frac{\cos B}{\sin A}.\]

So

\[\cos b = \frac{\cos 122.27^\circ}{\sin 52.4^\circ} \Rightarrow \cos b = -0.67388 \]

\[\Rightarrow b = 132.3674^\circ.\]

Lastly we find length \( a \) using

\[\cos a = \frac{\cos A}{\sin B}.\]

So

\[\cos a = \frac{\cos 52.4^\circ}{\sin 122.27^\circ} \Rightarrow \cos a = 0.74160 \]

\[\Rightarrow a = 43.8180^\circ. \quad [AP], \ [TT]\]
6) b) A spherical triangle’s angles must sum to > 180°. Since \( \angle C = 90^\circ \) and \( \angle A = \angle B \), both \( \angle A \) and \( \angle B \) must be > 45°.

Since \( a = b \), \( \cos c = \cos a \cos b \) is always non-negative. Therefore, by the spherical Pythagorean Theorem, \( \cos c \) is always non-negative \( \Rightarrow c \leq 90^\circ \). □ \([TT],[AP]\)

9) b) Using \( \cos c = \cos a \cos b \) and \( \cos B = \cos b \sin A \), we solve for \( \cos b \) in the second, and substitute it into the first and then simplify to get

\[
\sin A \cos c = \cos a \cos B. \quad \square \quad [TT],[AP]
\]

10) a) We first rearrange our spherical identity as follows:

\[
\sin B = \frac{\sin b}{\sin c}.
\]

Since \( \frac{\sin b}{\sin c} \to \frac{b}{c} \) as \( b, c \to 0 \), we have

\[
\sin B = \frac{b}{c}. \quad \square \quad [AP],[TT]
\]
2) Given the angles $A = 100^\circ$, $B = 69^\circ$, $C = 84^\circ$ we can use the law of cosines for angles to determine side $c$. Solving for $c$ we find

$$c = \cos^{-1} \left( \frac{\cos 84^\circ + \cos 100^\circ \cos 69^\circ}{\sin 100^\circ \sin 69^\circ} \right) = 87.36^\circ.$$ 

Rearranging the variables in the law of cosines for angles, we determine $a$ and $b$:

$$b = \cos^{-1} \left( \frac{\cos 69^\circ + \cos 100^\circ \cos 84^\circ}{\sin 100^\circ \sin 84^\circ} \right) = 69.67^\circ.$$ 

$$a = \cos^{-1} \left( \frac{\cos 100^\circ + \cos 69^\circ \cos 84^\circ}{\sin 69^\circ \sin 84^\circ} \right) = 98.43^\circ.$$ 

We determine each arc length by dividing them by $360^\circ$ and multiplying by the circumference of the circle. Since this applies to all three sides, we can instead use it on the sum of the sides $87.43^\circ + 69.67^\circ + 98.43^\circ = 255.46^\circ$, thus the perimeter is

$$\frac{255.46^\circ}{360^\circ} \cdot 2 \cdot \pi \cdot 10 \text{ in} = 44.58 \text{ in.} \quad [TT], [AP]$$
We will use \( \triangle ABC \) with \( \angle B \) at our initial point, \( \angle C \) at the south pole, and \( \angle A \) at our final position.

In Figure 6.1 the arc containing \( BC \) is E79° and the arc containing \( AC \) is E52°. Since we are crossing the meridian lines, we know \( \angle C = 27° \). Because \( c = 2060 \) nautical miles, then \( c = 34.33° \).

To find \( b \), the final latitude, we can use the sine law

\[
b = \sin^{-1} \left( \frac{\sin 34.33° \sin 50°}{\sin 27°} \right) = 72.10°
\]

Because this is the distance from the pole to \( A \), our latitude must be \( 90° - b = 17.90° \) S.

Our final bearing is \( A \), and we can use Napier’s second analogy to determine

\[
A = 2 \cdot \tan^{-1} \left( \frac{\sin \left( \frac{1}{2} \cdot (72.1° - 34.33°) \right)}{\sin \left( \frac{1}{2} \cdot (72.1° + 34.33°) \right) \tan \left( \frac{1}{2} \cdot (50° - 27°) \right)} \right) = 126.56°,
\]

so our bearing is \( 180° - A = 53.44° \).

Finally we can use Napier’s third analogy to find our initial latitude \( a \):

\[
a = 2 \cdot \tan^{-1} \left( \frac{\cos \left( \frac{1}{2} \cdot (50° + 27°) \right) \tan \left( \frac{1}{2} \cdot (72.1° + 34.33°) \right)}{\cos \left( \frac{1}{2} \cdot (50° - 27°) \right)} \right) = 93.77°.
\]

From this value our initial latitude can be determine using \( a - 90° = 3.77° \) N. [AP], [TT]
13) Construct the $\triangle ABC$ with $\angle A$ at Vancouver, $\angle B$ at Edmonton, and $\angle C$ at the north pole

where $\angle C = 9.6^\circ$, the difference between the meridians. We are given the latitude for the two cities: Vancouver= 49.3° N and Edmonton= 53.6° N, so we find $a = 36.4^\circ$ and $b = 40.7^\circ$. Using Napier’s first and second analogies we can construct a system of equations for $A + B$ and $A - B$ which will allow us to find both angles.

\[
A - B = 2 \cdot \tan^{-1} \left( \frac{\sin \left( \frac{1}{2} \cdot (36.4^\circ - 40.7^\circ) \right)}{\sin \left( \frac{1}{2} \cdot (36.4^\circ + 40.7^\circ) \right) \tan \left( \frac{1}{2} \cdot (9.6^\circ) \right)} \right) = -71.27^\circ,
\]

\[
A + B = 2 \cdot \tan^{-1} \left( \frac{\cos \left( \frac{1}{2} \cdot (36.4^\circ - 40.7^\circ) \right)}{\cos \left( \frac{1}{2} \cdot (36.4^\circ + 40.7^\circ) \right) \tan \left( \frac{1}{2} \cdot (9.6^\circ) \right)} \right) = 172.48^\circ.
\]

Combining (1) and (2): $A = 50.61^\circ$ and $B = 121.87^\circ$. We now use the law of sines to determine the distance between the cities

\[
c = \sin^{-1} \left( \frac{\sin 9.6^\circ \sin 36.4^\circ}{\sin 50.61^\circ} \right) = 7.35^\circ.
\]

To find the distance, knowing the radius of the earth is 6371 km,

\[
\frac{7.35^\circ}{360^\circ} \cdot 2 \cdot 6371 \text{ km} \cdot \pi = 817.28 \text{ km}. \ [TT], [AP]
\]
14) a) We begin with two cosine laws, one for $A$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

and another for $B$

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b.$$ 

We add them together to get

$$\cos A + \cos B = -\cos B \cos C - \cos A \cos C + \sin B \sin C \cos a + \sin A \sin C \cos b.$$ 

We simplify:

$$\cos A + \cos B = -\cos C (\cos B + \cos A) + \sin C (\sin B \cos a + \sin A \cos b)$$

and find

$$(\cos A + \cos B)(1 + \cos C) = \sin C (\sin B \cos a + \sin A \cos b) \tag{1}$$

From the Law of Sines we have $\sin B = m \sin b$ and $\sin A = m \sin a$. Substitute them into (1):

$$(\cos A + \cos B)(1 + \cos C) = \sin C (m \sin b \cos a + m \sin a \cos b).$$

From the sine addition law, we get

$$(\cos A + \cos B)(1 + \cos C) = m \sin C \sin(a + b). \quad [AP], [TT]$$

b) Taking

$$\frac{\sin A + \sin B}{\sin a + \sin b} = m$$

and dividing it by our result from a), we find

$$\frac{\sin A + \sin B}{\sin a + \sin b} \cdot \frac{1}{(\cos A + \cos B)(1 + \cos C)} = \frac{m}{m \sin C \sin(a + b)}.$$ 

Simplify:

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin(a + b)} \cdot \frac{1 + \cos C}{\sin C}.$$ 

c) Converting $\sin(a + b)$ into $\sin(2 \left(\frac{a+b}{2}\right))$ and $\sin C$ into $\sin(2 \left(\frac{C}{2}\right))$, we find

$$\frac{2 \sin \frac{A+B}{2} \cos \frac{A-2}{2}}{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}} = \frac{2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2} \left(2 \sin \frac{a+b}{2} \cos \frac{a+b}{2}\right)}.$$ 

d) Simplifying gives

$$\tan \frac{A+B}{2} \cot \frac{C}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}}. \quad [TT], [AP]$$